

Laplace and Fourier Transforms  
Lecture notes – Summary  
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# Laplace and Fourier Transforms

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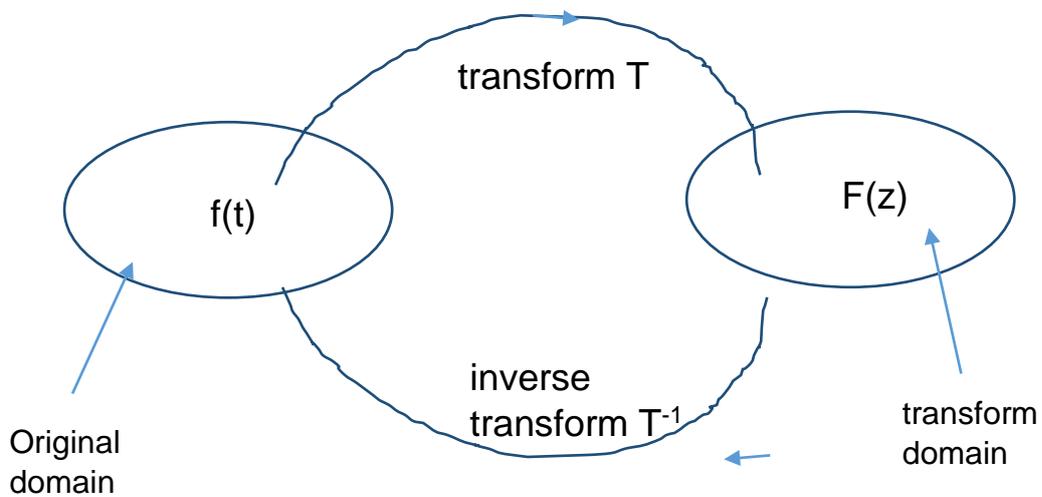
- 1.1 Definition
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# Chapter 1 – Transform theory

## 1.1 Definition of a transform

A transform is a mathematical mapping  $T$  from one domain into another. This mapping is accomplished by means of an expression that takes a function  $f$  of some variable(s)  $t$  usually real and projects it to a new function  $F=T(f)$  of a new variable  $z$  (often complex).

A transform  $T$  always has an inverse  $T^{-1}$  that will accomplish the inverse transform of  $F$  into the original function  $f$ , i.e.  $f = T^{-1}(F)$ .



## 1.2 Benefits of transforms

Transforms are used in engineering applications because they help in the design and analysis of systems and signals. They may also simplify some engineering problem solutions as we shall see with specific transforms. For example if we take a signal  $s(t)$  in its original domain (time), then we construct its Fourier transform in the frequency domain, the properties of this signal become much more visible.

Another example is that the convolution of two functions  $f(t)$  and  $g(t)$  in the original (time) domain is quite complex as it involves shift and integral operations. In the  $z$  transform domain however, it becomes a simple product of transformed functions  $F(z)$  and  $G(z)$  and we may then apply usual algebraic operations of product, division, etc.

In the 1960's, the discovery of the famous algorithm to compute the Fourier transform in a rapid manner known as "fast fourier transform" or FFT is considered as a breakthrough in the area of digital signal processing (DSP) especially in those days where computing power was a tiny fraction of what's available today to public.

### 1.3 Examples

Several transforms exist. The most popular ones are:

- Laplace Transform
- Fourier Transform
- Z Transform
- Hilbert transform
- Sine/cosine transform (a special form of Fourier)
- ...

## 1.4 Applications

For each particular transform, there are a number of applications. For example:

- Laplace transform: applications include electric circuit solutions, stability analysis, differential equations (see Chapter 2)
- Fourier transform: signal decomposition, compression, image analysis and filtering, image compression and synthesis like in MPEG2 (see Chapter 3)

## Chapter 2 – Laplace Transform

### 1.1 Definition

The Laplace transform  $F(s)$  of a the function  $f(t)$  in its general form is as follows:

$$LT(f(t)) = \int_{-\infty}^{+\infty} f(t)e^{-st} dt = F(s)$$

This form is called the bilateral transform. However, in practice,  $f(t)$  is zero for  $t < 0$  and thus the transform is limited to the region  $t \geq 0$  i.e.  $R^+$ :

$$F(s) = \int_0^{+\infty} f(t)e^{-st} dt$$

### 1.2 Main Properties

Let  $F(s)=LT\{f(t)\}$ ,  $G(s)=LT\{g(t)\}$ ,  $\alpha$  and  $\beta$  constants.

a. Linearity:

$LT\{\alpha f(t)\} = \alpha F(s)$  and  $LT\{f(t)+g(t)\} = F(s) + G(s)$  make the general rule:

$$LT \{ \alpha f(t) + \beta g(t) \} = \alpha F(s) + \beta G(s)$$

b. Scaling:

$$LT\{f(\alpha t)\} = \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$$

c. Shift:

$$LT\{f(t-t_0)\} = e^{-st_0} F(s)$$

d. Differentiation

$$LT\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(t=0)$$

e. Convolution:

$LT \{ f(t) * g(t) \} = F(s)G(s)$  where  $*$  is the convolution operation which is transformed by LT into a simple product.

### 1.3 Examples

**a.** Let  $f(t) = \lambda e^{-\lambda t}$  for  $t \geq 0$ . Find its Laplace transform.

We simply use the expression of  $L(f)$  and replace  $f(t)$  by  $\lambda e^{-\lambda t}$ :

$$F(s) = \int_0^{+\infty} \lambda e^{-\lambda t} e^{-st} dt = \lambda \int_0^{+\infty} e^{-(\lambda+s)t} dt = \frac{\lambda}{\lambda+s}$$

**b.** Let  $f(t) = te^{-\lambda t}$  for  $t \geq 0$ . Find its Laplace transform.

We simply use the expression of  $L(f)$  and replace  $f(t)$  by  $te^{-\lambda t}$ :

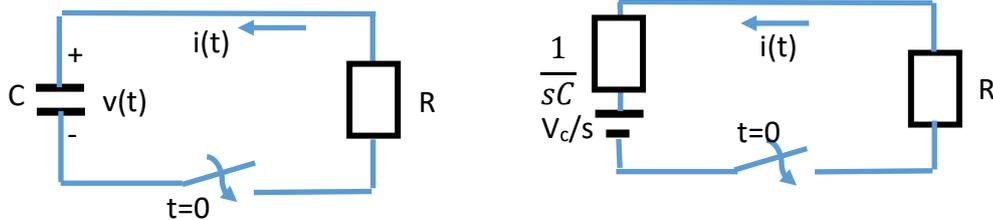
$$F(s) = \int_0^{+\infty} t e^{-\lambda t} e^{-st} dt = \int_0^{+\infty} t e^{-(\lambda+s)t} dt = \frac{1}{(\lambda+s)^2}$$

**c.** Basic functions such as impulse  $\delta(t)$ , step  $u(t)$ , sine, cosine, ... have known LT forms that can be found in all LT tables.

## 1.4 Applications

### A. Electric Circuits

Consider the simple RC circuit shown below (left). Suppose at  $t=0$  the switch is closed. If the capacitor has no charge then no current will flow. However if the capacitor is charged to a certain voltage  $V_c$  then at  $t=0$ :  $v(t=0)=V_c$  and a current  $i(t)$  will flow. Let us compute the expression of this current.



Two techniques may apply. In the first technique, we apply the usual circuit laws (Kirchoff and) for voltages and currents:

$$v(t) = - Ri(t) \text{ and } i(t) = C \frac{dv(t)}{dt} \text{ with the condition } v(t=0)=V_c.$$

Then we need to solve these equations and get the expression of  $i(t)$  and  $v(t)$ .

Another way is by using Laplace transform. The capacitor is replaced with a voltage source  $V_c/s$  and an impedance  $1/sC$ . Resistor  $R$  is left as is. Now we write the voltage equation around the circuit:

$$V_c/s - i(t)/sC - R i(t) = 0$$

$$\text{or } I(s) = \frac{V_c/R}{s + 1/RC}$$

and by inverse Laplace Transform:

$$i(t) = \frac{V_c}{R} e^{-t/RC} u(t)$$

## B. Differential Equations

Differential equations are of two types: ordinary differential equations (ODEs) and Partial Differential Equations (PDEs). ODEs are expressions where the unknowns are functions of one variable of the form  $f(x) = y$  and their derivatives  $y'$ ,  $y''$ ,  $y'''$  ...

PDEs are expressions where the unknowns are functions of two variables or more and their partial derivatives. PDEs will not be discussed in this chapter.

Using Laplace Transform to solve ODES is based on the main LT property:

$$\text{LT}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

which can be generalized to:

$$\begin{aligned}\text{LT}\left\{\left(\frac{df(t)}{dt}\right)'\right\} &= s \text{LT}\left\{\frac{df(t)}{dt}\right\} - \frac{df(t)}{dt}(t=0) = s [sF(s) - f(0)] - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

### Example 1:

Solve the equation:

$$y''(t) + k^2 y(t) = 0 \text{ with conditions } y(0)=A \text{ and } y'(0)=B.$$

Solution:

By  $\text{LT}[y''(t) + k^2 y(t)]$  we get:

$$s^2Y(s) - sy(0) - y'(0) + k^2 Y(s) = 0; \text{ then:}$$

$$Y(s) = \frac{As+B}{s^2+k^2}$$

which we can write as:

$$Y(s) = A \frac{s}{s^2+k^2} + \frac{B}{k} \frac{k}{s^2+k^2}$$

By looking for LT-1 in the table:

$$Y(t) = A \cos kt + \frac{B}{k} \sin kt.$$